This worksheet will work through the blowup of $\mathbb{C}^2$ (or $\mathbb{A}^2$) at the origin in several steps in order to hopefully inspire you to examine the blowup of $\mathbb{C}^3$ at the origin and/or your homework. Feel free to skip around.

Let e_1, e_2 be the standard basis vectors in $\mathbb{R}^2$, let $e_0 = e_1 + e_2$, and let Σ be the fan with maximal cones σ_2 generated by $\{e_0, e_1\}$ and σ_1 generated by $\{e_0, e_2\}$. (The unusual numbering is meant to agree with the higher dimensional case.)

(1) Define a map of toric varieties $\varphi\colon X_\Sigma \to \mathbb{A}^2$ using the two fans.

(2) Show that U_{σ_0} and U_{σ_2} are both isomorphic to $\mathbb{A}^2$.

(3) Describe the two restrictions $\mathbb{A}^2 \to \mathbb{A}^2$ of φ on coordinates.

(4) Which points of $\mathbb{A}^2$ are in the image of U_{σ_1} and which are in the image of U_{σ_2}?

(5) Find the (set-theoretic) preimage of each point in X_Σ under φ. [hint: There are two possibilities within each open.]

(6) The blowup is the closure of the graph $\Gamma(f)$ of the map $f\colon \mathbb{C}^2 \setminus \{(0,0)\} \to \mathbb{P}^1$ given by $f(a,b) = (a : b)$. Define a map $\psi\colon X_\Sigma \to \mathbb{C}^2 \times \mathbb{P}^1$ so that the following diagram commutes where the bottom arrow is inclusion:

$$\begin{array}{ccc} X_\Sigma & \xrightarrow{\psi} & \mathbb{A}^2 \times \mathbb{P}^1 \\ \varphi \downarrow & & \uparrow id \times f \\ \mathbb{A}^2 & \longleftarrow & \mathbb{A}^2 \setminus \{(0,0)\} \end{array}$$

(You could use fans for this but it's probably easier to define ψ on points directly.)

(7) Show that ψ is an isomorphism (or just a bijection) when restricted to $\Gamma(f)$.

(8) Describe the other points in the image of ψ.

(9) bonus: Let ρ be the ray generated by e_0. Show that the restriction of ψ to $V(\rho)$ (in the sense of the orbit-cone correspondence) is an isomorphism onto $\overline{\Gamma(f)} \setminus \Gamma(f)$.