(6) Given a point $(c:d)$ of $\mathbb{P}^1$, show that the preimage of $(c:d)$ under φ is also $\mathbb{P}^1$, at least on closed points. [hint: Start with the case $a = 0$.]

My map $\varphi\colon X \to \mathbb{P}^1$ is given by projection of the fan "downward." I will label the cones in the fan for $\mathbb{P}^1$ as follows:

$$\xleftarrow{\ \tau_1\ } \bullet \xrightarrow{\ \tau_0\ }$$

A point $(c:d)$ in $\mathbb{P}^1$ is contained in (at least) one of the open sets $U_{\tau_0} = \{(c:d) \mid d \neq 0\}$ and $U_{\tau_1} = \{(c:d) \mid c \neq 0\}$. Assume $(c:d)$ is of the second type, which is more difficult.

The open sets in X which map to U_{τ_1} correspond to the cones σ_1 and σ_2, which intersect in the ray ρ_2 spanned by $(-1, a)$. We have

- $U_{\sigma_1} = \operatorname{Spec} \mathbb{C}[x^a y, x^{-1}] \simeq \operatorname{Spec} \mathbb{C}[z, x^{-1}] \simeq \mathbb{A}^2$
- $U_{\sigma_2} = \operatorname{Spec} \mathbb{C}[x^{-a} y^{-1}, x^{-1}] \simeq \operatorname{Spec} \mathbb{C}[z^{-1}, x^{-1}] \simeq \mathbb{A}^2$
- $U_{\rho_2} = \operatorname{Spec} \mathbb{C}[x^a y, x^{-a} y^{-1}] \simeq \operatorname{Spec} \mathbb{C}[z, z^{-1}, x^{-1}] \simeq \mathbb{C}^* \times \mathbb{A}^1$
- $U_{\tau_1} = \operatorname{Spec} \mathbb{C}[x^{-1}] \simeq \mathbb{A}^1$

where I have taken $z = x^a y$ throughout. Note that x^{-1} is algebraically independent from both z and z^{-1} so this is not misleading. The map φ on the open cover of X corresponds to the inclusion of $\mathbb{C}[x^{-1}]$ into each of the three rings.

In U_{τ_1} the point $(c:d) = \left(1 : \frac{c}{d}\right)$ corresponds to the maximal ideal $\left\langle x^{-1} - \frac{c}{d} \right\rangle$. The extension of this ideal to each of the rings above has the same generator. Thus the preimage of $(c:d)$ under φ consists of the sets

- $\operatorname{Spec}\left(\mathbb{C}[z, x^{-1}] / \left\langle x^{-1} - \frac{c}{d} \right\rangle\right) \simeq \operatorname{Spec} \mathbb{C}[z] \simeq \mathbb{A}^1$
- $\operatorname{Spec}\left(\mathbb{C}[z^{-1}, x^{-1}] / \left\langle x^{-1} - \frac{c}{d} \right\rangle\right) \simeq \operatorname{Spec} \mathbb{C}[z^{-1}] \simeq \mathbb{A}^1$

identified along $\operatorname{Spec} \mathbb{C}[z, z^{-1}]$, which is indeed $\mathbb{P}^1$.