We are going to construct an affine toric variety from a *strongly convex rational polyhedral cone.*

Recall that the affine semigroup ring of our variety will be a subring of $\mathbb{C}[M]$, the ring of a torus T with character group M. However we start with the dual $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$, which is also isomorphic to $\mathbb{Z}^n$ for some n but whose generators we will not think of as characters.

A *convex polyhedral cone* σ in $N \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^n$ is the positive span of a finite set of vectors $u_1, \ldots, u_s \in \mathbb{R}^n$: $\sigma = \{\sum \lambda_i u_i \mid \lambda_i \in \mathbb{R}, \lambda_i \geq 0\}$.

(1) Draw the cone spanned by $(0,1)$ and $(-1,2)$ in $\mathbb{R}^2$.

(2) Show that σ is always convex.

(3) A *rational* polyhedral cone is one spanned by some $u_1, \ldots, u_s \in N$ (meaning that the vectors have integer entries). Describe a cone which is not rational.

(4) A *strongly convex* polyhedral cone is one with $\sigma \cap (-\sigma) = \{0\}$. Describe a cone which is not strongly convex.

Next week we will learn how to take the *dual* of a cone σ in $N \otimes_{\mathbb{Z}} \mathbb{R}$ to get a cone $\sigma^\vee$ in $M \otimes_{\mathbb{Z}} \mathbb{R}$. (Remember that these vector spaces are both isomorphic to $\mathbb{R}^n$.)

(5) Assume that $\sigma^\vee$ is a rational polyhedral cone and show that $\sigma^\vee \cap M$ (the set of elements in $\sigma^\vee$ with integral entries) forms a semigroup.

(6) The dual of the cone from (1) is spanned by $(-1, 0)$ and $(2, 1)$. Find generators of the semigroup.