On the last worksheet you found a bijection between $\mathbb{Z}^n$ and the characters of a torus $T = \mathrm{Spec}(\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}])$. The textbook uses the notation χ^m for the character $\chi^m : (\mathbb{C}^*)^n \to \mathbb{C}^*$ corresponding to $m \in \mathbb{Z}^n$.

You might need to adjust your bijection and/or the isomorphism you define below in order to make the following properties true.

(1) Show that multiplying two characters (in the sense of multiplying functions) corresponds to adding vectors in $\mathbb{Z}^n$.

(2) Define an isomorphism between $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and the affine semigroup ring $\mathbb{C}[\mathbb{Z}^n]$.

(3) Show that a matrix $\mathbb{Z}^s \to \mathbb{Z}^n$ induces a map of affine semigroup rings $\mathbb{C}[\mathbb{Z}^s] \to \mathbb{C}[\mathbb{Z}^n]$.

(4) Show that your isomorphism from (2) is canonical in the sense that the character $\chi^m \colon (\mathbb{C}^*)^n \to \mathbb{C}^*$ is induced by the map $\mathbb{Z} \to \mathbb{Z}^n$ given by the vector m. (First get a map of affine semigroup rings, then a map of affine varieties, and then a map on closed points.)

(5) Try to explain why the textbook uses the same notation for the character χ^m corresponding to $m \in \mathbb{Z}^n$ and the element χ^m of the affine semigroup ring $\mathbb{C}[\mathbb{Z}^n]$.

You have shown that $T \simeq \operatorname{Spec}(\mathbb{C}[M])$ where M is the (affine semi)group of characters of T under multiplication. We know that M is isomorphic to $\mathbb{Z}^n$, but we will use $\mathbb{C}[M]$ to avoid choosing a basis for it. Changing bases would correspond to a linear change of variables on the ring $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.