

Section 1.0 of the textbook describes the *product* of two affine varieties $X = \text{Spec } R$ and $Y = \text{Spec } S$ as a special case of a more general construction. For now we will take $\text{Spec}(R \otimes_{\mathbb{C}} S)$ as the definition of $X \times Y$.

Suppose $R = \mathbb{C}[x_1, \dots, x_n]/\mathfrak{p}$ where \mathfrak{p} corresponds to the point (a_1, \dots, a_n) in \mathbb{C}^n and $S = \mathbb{C}[y_1, \dots, y_m]/\mathfrak{q}$ where \mathfrak{q} corresponds to the point (b_1, \dots, b_m) in \mathbb{C}^m .

(1) Construct a map $X \times Y \rightarrow \text{Spec}(\mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_m])$.

(2) What is the image of your map from (1) geometrically?

(3) Which ideal in $\mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_m]$ defines the image of $X \times Y$? How does this ideal relate to \mathfrak{p} and \mathfrak{q} ?

(4) Describe a point in $\text{Spec } \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_m]$ not of the form you described in (3).

(5) How might you imagine your point from (4) geometrically in \mathbb{C}^{n+m} ?

(6) Define $f: \mathbb{C}^n \rightarrow \mathbb{C}^m$ by $a \mapsto (f_1(a), \dots, f_m(a))$ where f_1, \dots, f_m are polynomials in n variables. Describe the corresponding map of rings $\mathbb{C}[y_1, \dots, y_m] \rightarrow \mathbb{C}[x_1, \dots, x_n]$.

(7) What does your map in (6) do to non-maximal ideals (“non-closed points”) in $\text{Spec } \mathbb{C}[x_1, \dots, x_n]$?

(8) Describe the map $\mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}^n$ given by $(a, b) \mapsto a + b$ as a map of rings.