Lemma (silly cone fact)**.** *Let C be a cone generated by $u_1, \ldots, u_t \in \mathbb{R}^n$. If c is in the (relative) interior of C, meaning that $\langle m, c\rangle > 0$ for all $m \in C^\vee$, then $c = \sum_{i=1}^t c_i u_i$ for some $c_1, \ldots, c_t > 0$.*

Proof. Induct on t. By replacing $\mathbb{R}^n$ with a linear subspace we can assume that $\dim C = n$, so that $t \geq n$. If $t = n$ then some change of coordinates sends $u_1, \ldots, u_n$ to the standard basis vectors in $\mathbb{R}^n$. Then c is in the interior of C if and only if it lies on no coordinate plane if and only if it has all positive coordinates.

Assume the lemma holds for the cone B generated by $u_1, \ldots, u_t$ and consider c in the interior of the cone $C \supseteq B$ generated by $u_1, \ldots, u_t, u_{t+1}$. Let $m_1, \ldots, m_k, \ldots, m_\ell$ be the generators of $B^\vee$ so that $m_1, \ldots, m_k \in C^\vee \subseteq B^\vee$ and $m_{k+1}, \ldots, m_\ell \notin C^\vee$. Write $c = \sum_{i=1}^{t+1} c_i u_i$ for $c_0, \ldots, c_t, c_{t+1} \geq 0$. If $c_i > 0$ for all i then we are done.

Otherwise let $c' = c - c_{t+1}u_{t+1}$, so that $c' \in B$. If $c_{t+1} > 0$ and c' is in the interior of B then we are done by the induction hypothesis. If not then compare $\langle m_i, c'\rangle$ and $\langle m_i, u_{t+1}\rangle$ for $1 \leq i \leq \ell$. Recall that $\langle m_i, u_{t+1}\rangle \geq 0$ for $1 \leq i \leq k$ and $\langle m_i, u_{t+1}\rangle < 0$ for $k < i \leq \ell$.

case 1: $c_{t+1} = 0$

Then $c' = c$ so $\langle m_i, c\rangle > 0$ for all i. Choose $d > 0$ so that $d\langle m_i, c\rangle > \langle m_i, u_{t+1}\rangle$ for $1 \leq i \leq k$. This implies that $c - \frac{1}{d}u_{t+1}$ is in the interior of B, so we are done.

case 2: $c_{t+1} \neq 0$

Then $\langle m_i, c'\rangle \geq 0$ for all i because $c' \in B$, but also $\langle m_i, c'\rangle > 0$ for $k < i \leq \ell$ because subtracting u_{t+1} from c can only increase these values. Similarly for each $1 \leq i \leq k$ we have either $\langle m_i, c'\rangle = 0$ or $\langle m_i, u_{t+1}\rangle = 0$ but not both. Choose $d > 0$ so that $d\langle m_i, c'\rangle > -\langle m_i, u_{t+1}\rangle$ for $k < i \leq \ell$ and $dc_{t+1} > 1$. This implies that $c' + \frac{1}{d}u_{t+1}$ is in the interior of B where $c' + \frac{1}{d}u_{t+1} = c - \left(c_{t+1} - \frac{1}{d}\right)u_{t+1}$ for $c_{t+1} - \frac{1}{d} > 0$.

□

Corollary (probably helpful for homework)**.** *Let C be a rational polyhedral cone and $B \subseteq C$ a rational polyhedral subcone such that for all $c, c' \in C$ we have $c + c' \in B$ if and only if $c \in B$ and $c' \in B$. If B contains c in the (relative) interior of C then B contains all of C.*

Proof. Let $u_1, \ldots, u_t$ be the generators of C and write $c = \sum_{i=1}^t c_i u_i$ with $c_i > 0$ for all i, which is possible by the previous lemma. Then $c_i u_i \in B$ for all i by assumption so $u_i \in B$ for all i because B is a cone. □