Let T be a torus with characters M and dual lattice N, let Σ be a fan in $N \otimes_{\mathbb{Z}} \mathbb{R}$, and let X_Σ be the toric variety of Σ, which is obtained from open sets $U_\sigma = \operatorname{Spec} \mathbb{C}[\sigma^v \cap M]$ for $\sigma \in \Sigma$.

For each $\sigma \in \Sigma$ define:

- γ_σ to be the point in U_σ corresponding to the semigroup homomorphism $\sigma^\vee \cap M \to \mathbb{C}$ given by $m \mapsto 1$ if $m \in \sigma^\vee \cap \sigma^\perp$ and $m \mapsto 0$ otherwise
- $O(\sigma)$ to be the orbit of γ_σ under the action of T
- $V(\sigma)$ to be the Zariski closure of $O(\sigma)$ meaning either
  - the smallest Zariski closed subset X_Σ containing $O(\sigma)$, or equivalently
  - the smallest subset $V \subseteq X_\Sigma$ containing $O(\sigma)$ so that $V \cap U_{\sigma'}$ is the vanishing of some ideal $I_\sigma \subseteq \mathbb{C}[(\sigma')^\vee \cap M]$ for each $\sigma' \in \Sigma$

Lemma (3.2.4)**.** *Write N_σ for the subgroup of N spanned by the elements of some cone σ. Then $\sigma^\perp \cap M$ is a subgroup of M and its dual is (naturally) isomorphic to N/N_σ.*

Proof. If $m, m' \in \sigma^\perp \cap M$ then $\langle m + m', u\rangle = \langle m, u\rangle + \langle m', u\rangle = 0$ and $\langle -m, u\rangle = -\langle m, u\rangle = 0$ for all $u \in \sigma$ so $\sigma^\perp \cap M$ is closed under addition and inverses. It also contains 0.

Considered as a map $N \to \mathbb{Z}$, an element $m \in M$ is in $\sigma^\perp$ if and only if $\langle m, u\rangle = 0$ for all $u \in \sigma$ if and only if $\langle m, u\rangle = 0$ for all u in the subgroup generated by σ if and only if m factors through $N/N_i \to \mathbb{Z}$. Thus m is in the dual of N/N_i exactly when $m \in \sigma^\perp$. □

Lemma. *If M and L are $\mathbb{Z}$-modules and M is free of finite rank then $M^* \otimes_{\mathbb{Z}} L = \operatorname{Hom}_{\mathbb{Z}}(M, L)$ where M^* is the dual $\operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ of M. (This is also true over other rings.)*

Proof. There is a bilinear map $M^* \times L \to \operatorname{Hom}_{\mathbb{Z}}(M, L)$ where the image of the pair (φ, l) sends $m \in M$ to $\varphi(m)l$, the product of l and the integer $\varphi(m)$ in the $\mathbb{Z}$-module L. Thus there is a $\mathbb{Z}$-linear map $M^* \otimes_{\mathbb{Z}} L \to \operatorname{Hom}_{\mathbb{Z}}(M, L)$. It is an isomorphism because its source and target are both isomorphic to L^r for r the rank of M. □

Lemma (3.2.5)**.** *Fix $\sigma \in \Sigma$. Let O' be the set of semigroup homomorphisms $\gamma \colon \sigma^\vee \cap M \to \mathbb{C}$ so that $\gamma(m) \neq 0$ exactly when $m \in \sigma^\perp \cap M$.*

(1) The closed points T' of the torus $\operatorname{Spec} \mathbb{C}[\sigma^\perp \cap M]$ are in bijection with elements of O'.

(2) The orbit $O(\sigma)$ in U_σ is O'.

Proof. The space $\operatorname{Spec} \mathbb{C}[\sigma^\perp M]$ is a torus because $\sigma^\perp \cap M$ is a subgroup of M, which must also be a lattice. By Proposition 1.3.1 the closed points of $\operatorname{Spec} \mathbb{C}[\sigma^\perp \cap M]$ are semigroup homomorphisms $\sigma^\perp \cap M \to \mathbb{C}$, which are the same as group homomorphisms $\sigma^\perp \cap M \to \mathbb{C}^*$. By Exercise 3.2.5 the set of such homomorphisms is isomorphic to O'.

By Exercise 1.3.1 $(t \cdot \gamma_\sigma)(m) = \chi^m(t)\gamma(m)$ for $t \in T$ and $m \in \sigma^\vee \cap M$. Since $\chi^m(t) \neq 0$ we have $t \cdot \gamma_\sigma \in O'$ and thus $O(\sigma) \subset O'$. There is a surjection $N \to N/N_\sigma$, which remains surjective after applying $\otimes_{\mathbb{Z}} \mathbb{C}^*$. Note that

$$N \otimes_{\mathbb{Z}} \mathbb{C}^* = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*)$$

by the previous lemma because N is the dual of M. Similarly

$$(N/N_\sigma) \otimes_{\mathbb{Z}} \mathbb{C}^* = \operatorname{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, \mathbb{C}^*)$$

because N/N_σ is the dual of $\sigma^\perp \cap M$ by Lemma 3.2.4. Thus there is a surjection $T \to T'$ (which respects their group structures). The group T' acts transitively on itself, meaning that for each $\gamma \in T'$ there exists some $t' \in T'$ so that $t' \cdot \gamma_\sigma = \gamma$. Hence there also exists $t \in T$, mapping to t', so that $t \cdot \gamma_\sigma = \gamma$ and $T' \subset T \cdot \gamma_\sigma = O(\sigma)$.

Thus we have $T' \subset O(\sigma) \subset O'$ where $T' = O'$ by (1). □

Lemma (Exercise 3.2.6(c))**.** *If τ is a face of $\sigma' \in \Sigma$ then $V(\tau) \cap U_{\sigma'}$ is defined by the ideal*

$$I = \left\langle \chi^m : m \in (\sigma')^\vee \cap M \setminus \tau^\perp \right\rangle$$

in $\mathbb{C}[(\sigma')^\vee \cap M]$.

Lemma. *Suppose γ is a semigroup homomorphism representing a point in U_σ for some $\sigma \in \Sigma$ and let O be its orbit. Then the set $\{m \in \sigma^\vee \cap M : \gamma(m) \neq 0\}$ is equal to $\sigma^\vee \cap \tau^\perp \cap M$ for some face τ of σ and O is contained in U_τ.*

Proof. By Exercise 3.2.6(a) the given set is the intersection of M with a face of $\sigma^\vee$, and each face of $\sigma^\vee$ is of the form $\sigma^\vee \cap \tau^\perp$ for a face τ of σ by Proposition 1.2.10. Then by the proof of Proposition 1.3.16 there is some $m \in \sigma^\vee \cap M$ so that

$$\sigma^\vee \cap M + \mathbb{Z}(-m) = \tau^\vee \cap M.$$

This equality implies that $m \in \tau$ and $-m \in \tau$ so $m \in \sigma^\vee \cap \tau^\perp \cap M$. Thus γ is nonzero on m by choice of τ, so γ can be extended to $-m$ and thus to $\tau^\vee \cap M$, meaning that γ is a point of U_τ. Since U_τ is closed under the action of T we must have $O = T \cdot \gamma =\subseteq U_\tau$. □

Theorem (3.2.6)**.** *Use the notation above.*

(a) There is a bijection between the cones in Σ and the orbits of T in X_Σ given by

$$\sigma \mapsto O(\sigma).$$

(b) For each $\sigma \in \Sigma$

$$\dim O(\sigma) = \dim X_\Sigma - \dim \sigma.$$

(c) For each $\sigma \in \Sigma$ the affine open

$$U_\sigma = \bigcup_\tau O(\tau)$$

over all faces τ of σ.

(d) For each $\tau \in \Sigma$ the subvariety

$$V(\tau) = \bigcup_\sigma O(\sigma)$$

over all cones $\sigma \in \Sigma$ containing τ as a face.

Proof. (a) Fix an orbit O of T and let σ be the minimum cone in Σ satisfying $O \subseteq U_\sigma$ (which exists because the U_σ are closed under intersection). We will show that $O = O(\sigma)$. This implies directly that $\sigma \mapsto O(\sigma)$ is surjective, but also implies that it is injective because $O(\sigma)$ and $O(\sigma')$ cannot be equal if they are minimally contained in distinct opens U_σ and $U_{\sigma'}$.

Choose a semigroup homomorphism $\gamma \in O \subseteq U_\sigma$. By the previous lemma there is some face τ of σ so that γ is nonzero exactly on $\sigma^\vee \cap \tau^\perp$, and γ is actually contained in $U_\tau \subseteq U_\sigma$. However $\gamma \in U_\tau$ implies $O = T \cdot \gamma \subseteq U_\tau$, so by choice of σ we must have $\tau = \sigma$. Therefore O is exactly the set O' appearing in Lemma 3.2.5, which is shown there to be equal to $O(\sigma)$.

(c) Fix $\sigma \in \Sigma$. Since U_σ is closed under the action of T it must be a union of orbits, so we must only determine which appear. If $\gamma \in U_\sigma$ then γ is nonzero exactly on $\sigma^\vee \cap \tau^\perp$ for a face τ of σ by the previous lemma, so by Lemma 3.2.5 the point γ is contained in $O(\tau)$ for some face τ of σ, as desired. Conversely $O(\tau) \subseteq U_\tau \subseteq U_\sigma$ for each τ.

(b) By the proof of Lemma 3.2.5 the orbit $O(\sigma)$ is a torus with characters dual to N/N_σ. Thus its dimension is equal to $\dim(N/N_\sigma) = \dim N - \dim N_\sigma = \dim X_\Sigma - \dim \sigma$.

(d) Fix $\tau \in \Sigma$. It is enough to verify the equality of sets after intersecting with the open cover $\{U_{\sigma'} : \sigma' \in \Sigma\}$. Thus we will show that

$$V(\tau) \cap U_{\sigma'} = \bigcup_{\sigma} O(\sigma)$$

where we restrict the union to cones σ containing τ as a face such that $O(\sigma)$ is also contained in $U_{\sigma'}$. By (c) these are exactly the σ with τ a face of σ and σ a face of σ'. By Exercise 3.2.6(c) (restated above due to the typo) the ideal defining $V(\tau) \cap U_{\sigma'}$ is $I = \langle \chi^m : m \in (\sigma')^\vee \cap M \setminus \tau^\perp \rangle$.

Let $\gamma \in O(\sigma)$ for τ a face of σ a face of σ', considered as a semigroup homomorphism $(\sigma')^\vee \cap M \to \mathbb{C}$. (Exercise 3.2.5 helps us see that it doesn't matter whether we use σ or σ' for the codomain of γ.) Then γ is zero outside of $(\sigma')^\vee \cap \sigma^\perp$. Since $\tau \subseteq \sigma$ we have $\tau^\perp \supseteq \sigma^\perp$ and $(\sigma')^\vee \cap M \setminus \tau^\perp \subseteq (\sigma')^\vee \cap M \setminus \sigma^\perp$. Hence for $\chi^m \in I$ we also have $m \in (\sigma')^\vee \cap M \setminus \sigma^\perp$ so $\gamma(m) = 0$. By Exercise 1.3.1 plugging m into γ is the same as evaluating χ^m at the point corresponding to γ, so I vanishes on γ and thus $O(\sigma) = T \cdot \gamma \subseteq V(\tau) \cap U_{\sigma'}$ because it is closed under the action of T.

Conversely let $\gamma \in V(\tau) \cap U_{\sigma'}$. By the previous lemma γ vanishes outside of $(\sigma')^\vee \cap \sigma^\perp$ for some face σ of σ' and $\gamma \in O(\sigma)$ by Lemma 3.2.5. Since I vanishes on γ we must have $(\sigma')^\vee \cap M \setminus \tau^\perp \subseteq (\sigma')^\vee \cap M \setminus \sigma^\perp$, so $\tau^\perp \supseteq \sigma^\perp$ and $\tau \subseteq \sigma$. Since $V(\tau) \cap U_{\sigma'}$ is nonempty it must contain a point of $O(\tau)$, so τ is a face of σ' and thus also a face of σ.

$\square$